2.8 Upper bound calculation

- **Determination of mechanism for plastic collapse**

(破壊のメカニズムの決定)

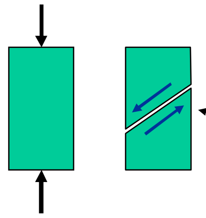
(**permissible velocity field**)

- *direction of slip plane*
- *pattern (shape) of slip plane*

ground(continuum)

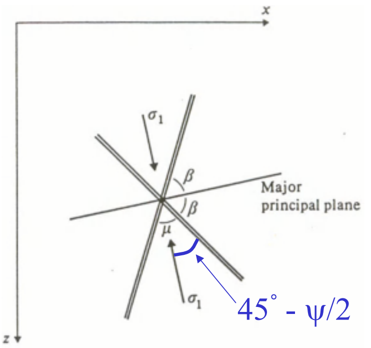
↑ →slip plane →**collapse mechanism**

external load



simple mechanism of plastic collapse

2.8.1 Direction of slip plane



- **in case of continuous strain**

slip plane

↓

zero-extension line (p7-10) $\beta = 45^\circ + 1/2\psi$, $\mu = 90^\circ - \psi$ **(31)**

- **in case rigid body**

failure pattern ⇔ relative displacement on slip planes

<**discontinuity：不連続面**>

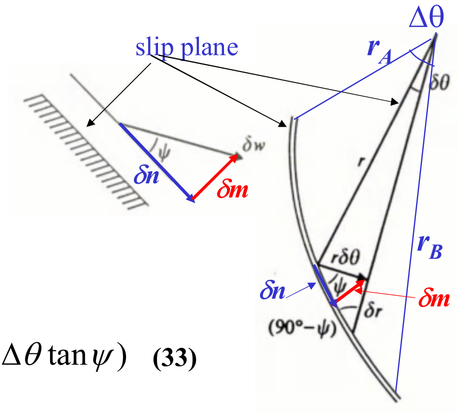
no restriction of the μ angle but in order to obtain ***a good mechanism giving good bound value***, the μ value is better to satisfy the above condition

$\mu = 90^\circ - \psi$

for undrained condition($\psi=0$) : $\mu = 90^\circ$

for drained condition($\psi = \phi'$): $\mu = 90^\circ - \phi'$ **(32)**

2.8.2 Pattern of slip planes



straight lines and curved lines

- possible mechanism

(= permissible velocity field)

-for **drained condition**: (ψ≠0)

$$\frac{\delta n}{\delta m}=\frac{dr}{r\,d\theta}=\tan\psi,\quad r_B/r_A=\exp(\Delta\theta\tan\psi)\qquad \textbf{(33)}$$

straight lines + log spiral

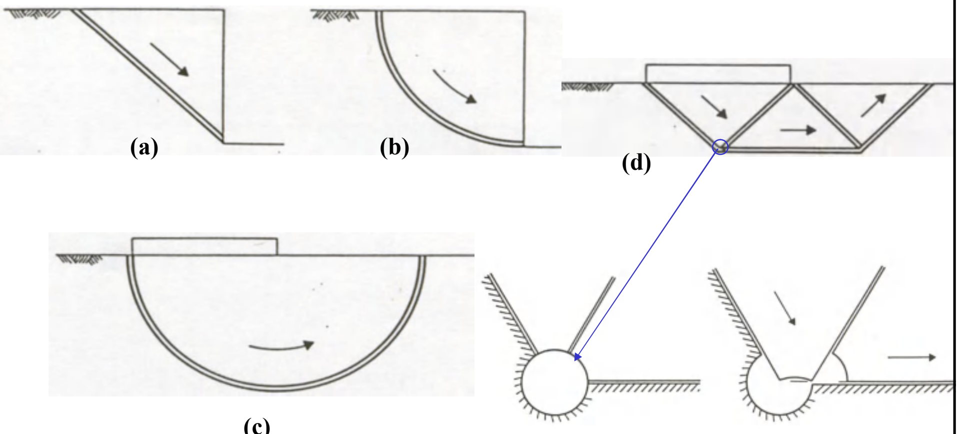
-for **undrained condition**: (ψ=0)

$$r_B/r_A=\exp(0)=1\qquad \textbf{(34)}$$

straight lines + circular arcs

Fig.4.12(p121) are all permissible mechanisms with rigid block.
Fig.4.12 (d) also compatible (referring Fig.4.13)

Possible mechanism of plastic collapse with rigid blocks



compatibility of displacement at the junction of three slip planes

2.8.3 Energy calculation

upper bound calculation ⇔ making energy equation

Two types of energy due to plastic failure

-increment of work done by **external loads: dE**

(**external work:外部仕事**)

including collapse load or height

-increment of work done by **internal stresses: dW**

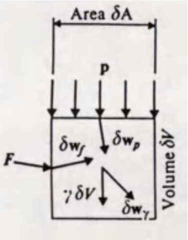
(**internal dissipation ： 内部消散**)

Energy equation:

$$\delta E = \delta W \longrightarrow f\,(\text{collapse load or height})$$

---

(1) External work: δE



Undrained

Undrained condition:

stress => total stress

displacement by body force

$$\delta E = \sum \delta \underline{\mathbf{w}_f} \cdot \mathbf{F} + \int_A \delta \underline{\mathbf{w}_p} \cdot \mathbf{p}\, dA + \int_V \delta \underline{\mathbf{w}_\gamma} \cdot \boldsymbol{\gamma}\, dV \qquad (4.30)$$

displacement by Force F

displacement by pressure p

vector inner products

F: point forces acting boundary surface

p: boundary stresses

γ: body forces (self weight: γ_{sat})

below ground water level

above GWL

Drained condition:

stress => effective stress **p=> p'=p-u, γ=>** $\gamma_{sat}-\gamma_w=\gamma'$ **or** γ_t

$$\delta E = \sum \delta \mathbf{w}_f \cdot \mathbf{F} + \int_A \delta \mathbf{w}_p \cdot (\mathbf{p} - \mathbf{u})\, dA + \int_V \delta \mathbf{w}_\gamma \cdot (\boldsymbol{\gamma} - \boldsymbol{\gamma}_w)\, dV \qquad (4.35)$$

(2) Internal dissipation: δW

Perfect plastic material: **elastic strain: $\delta\varepsilon_e = 0$** at failure

increment of work done by the internal stresses is **completely dissipated in plastic distortion** => ***thermal energy***

distortion with continuous strain: δW= ***stress*** x ***strain*** x δV,

discontinuity (2D problem)=> ***force*** x ***displacement***

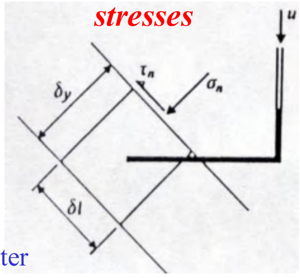
undrained condition: ***displacement= relative displacement along the slip***

For 2D condition

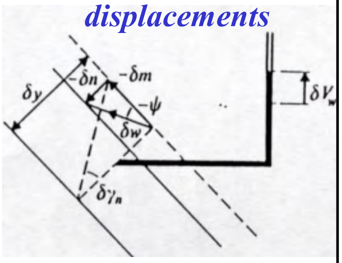
2D: per unit width in the direction of σ_2

$$\delta W = 1/V\left[-\delta n(\sigma_n \cdot \delta l) - \delta m(\tau_n \cdot \delta l) - u \cdot \underline{\delta V_w}\right] \quad (4.36)$$

volume of squeezed water by pore pressure u



stresses



displacements

stress and displacements across a slip plane

here

$$V = \delta y \delta l, \quad \delta\varepsilon_v = -\frac{\delta n}{\delta y}, \quad \delta\gamma_n = -\frac{\delta m}{\delta y}$$

compression:+, counter clockwise:+

in case of saturation: $-\delta V = \delta V_w$, $\varepsilon_v = -\delta V/V$

$$\delta W = \left[\sigma_n \delta\varepsilon_v + \tau_n \delta\gamma_n - u \cdot \delta\varepsilon_v\right]$$
$$= \left[\sigma'_n \delta\varepsilon_v + \tau'_n \delta\gamma_n\right] \quad (4.37)$$

cont'd.

at failure $\tau'_n = \sigma'_n \tan\phi'$, $\tan\psi = -\dfrac{\delta\varepsilon_n}{\delta\gamma_n}$

eq.(4.37) => $\delta W = \tau'_n \cdot \delta\gamma_n \left[1 - \dfrac{\tan\psi}{\tan\phi'}\right]$ (4.38)

<u>for drained conditions</u>, like sand, normality condition (associated flow rule) gives $\psi=\phi'$. Then eq(4.38) becomes

$$\boxed{\delta W = 0 \qquad (4.39)}$$

only δE calculation is needed.

<u>for undrained conditions</u> ($\delta V=0$), $\delta n=\psi=0$, $\delta w=-\delta m$

internal dissipation in unit length

$$\delta W = 1/\delta l\left[-\delta m(\tau_n \cdot \delta l)\right] \qquad (4.40)$$

independence of thickness of slip plane δy

for complete collapse mechanism with $\tau_{nf}=c_u$

$$\boxed{\delta W = \sum c_u \cdot L \cdot \delta w \qquad (4.41)}$$

no internal stress is included

length of slip plane, relative displacement

upper bound theorem

2.8.4 Failure mechanisms:

To obtain upper bound (>true collapse load) using upper bound theory, **kinematically admissible failure mechanism** (*permissible velocity filed*) is only required. But from unrealistic mechanism, unrealistic load which is far greater than the true collapse load is obtained. **Good mechanism close to real failure pattern can give good solution, <u>which can be used in design under F_s</u>**.

U conditions: $\phi=0$ material => $\delta W = \delta E$

D conditions: ϕ material ($\psi=\phi'$) => $\delta W(=0) = \delta E$

Energy calculation: Energy = *force* x *displacement* or *stress* x *strain*

displacements <u>in the direction of</u> forces:

-boundary force ⇔ boundary surface displacement;

-body force (gravity) ⇔ vertical displacement of the center of gravity;

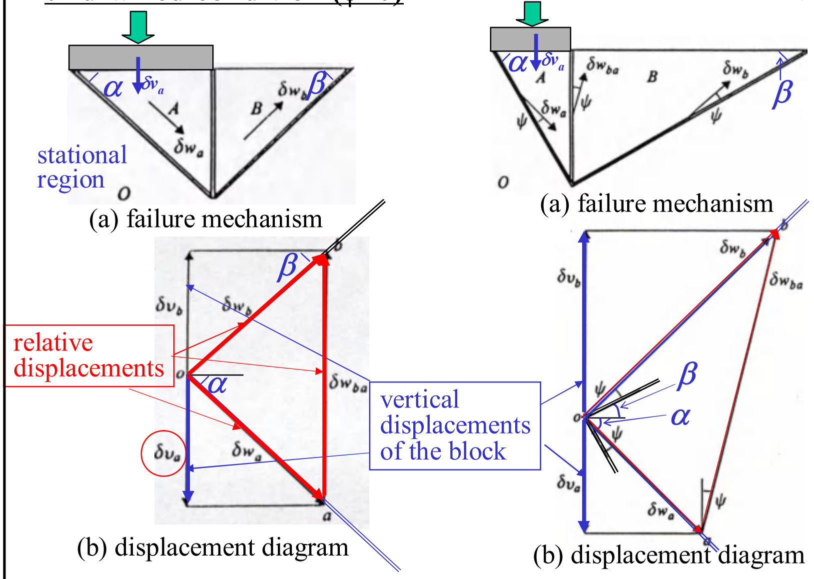
-shear force ⇔ relative displacement along slip surface

Displacement Diagrams can be used to obtain these displacements.

(1) Displacement diagram

undrained condition (ϕ=0)

drained condition (ϕ'=ψ)



stational region

relative displacements

vertical displacements of the block

(a) failure mechanism

(a) failure mechanism

(b) displacement diagram

(b) displacement diagram

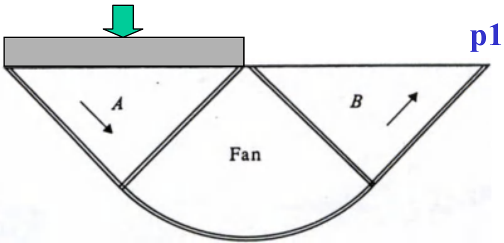
---

(2) Curved failure surface: Fan

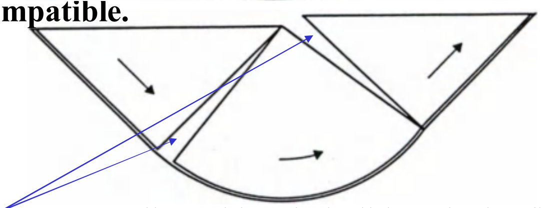
possible failure surfaces: straight line + curve

p127

common mechanism:
for clay(ϕ_u=0)

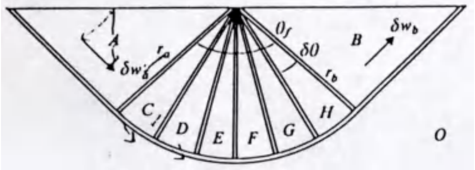


If three parts are rigid,
the mechanism is not compatible.



kinematically **gaps** are not allowed in admissible velocity field.

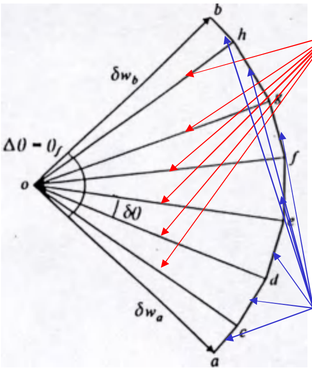
Fan for undrained condition (ϕ=0)



(a) compatible mechanism

$$\delta w_a = \delta w_b = \delta w$$

$$\delta W = \sum c_u R\left(\underline{\delta w \delta\theta}\right) + \sum c_u \left(R\delta\theta\right)\underline{\delta w} \quad (4.45)$$



(b) displacement diagram

relative displacements betw. O and arc portion

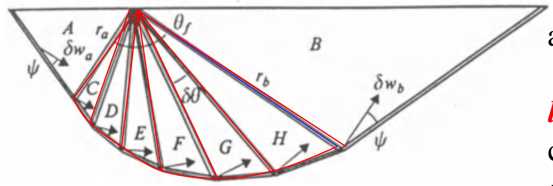
relative displacements at radial slip lines

at limit: δθ=>0

$$\delta W = \int_0^{\theta_f = \Delta\theta} 2c_u R \delta w \delta\theta \quad (4.46)$$

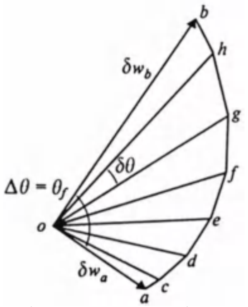
$$\delta W = 2c_u R \Delta\theta \delta w \quad (4.47)$$

Fan for drained condition (ϕ’=ψ>0)



(a) compatible mechanism

at limit: δθ=>0
curved slip plane becomes ***log spiral*** and corresponding displacement diagram are also ***log spiral***.
see next two pages



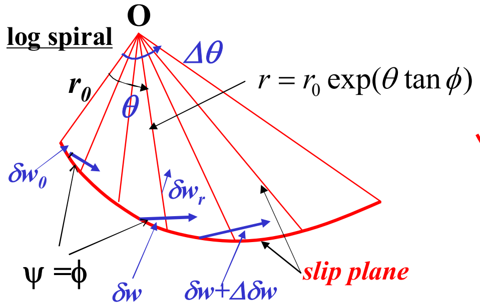
(b) displacement diagram

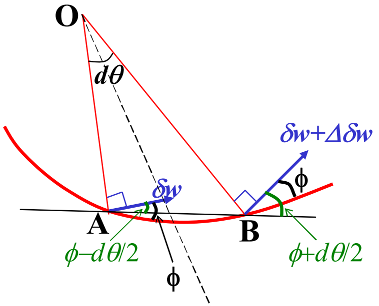
$$r_b = r_a \exp(\theta_f \tan\psi) \quad (4.43)$$

$$\delta w_b = \delta w_a \exp(\theta_f \tan\psi) \quad (4.44)$$

note: vertical components of displacement are only used for the energy calculation on external work due to self-weight for ϕ’=ψ condition.

Additional explanation on displacement diagram of log-spiral





With the no lap and no gap conditions between two rigid block, the velocity components of *δw and δw+Δδw* should be the same in the average direction (line AB). Hence,

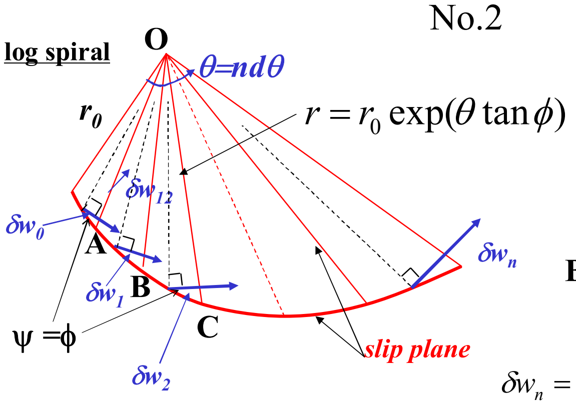
$$\delta w\cos\left(\phi-\frac{d\theta}{2}\right)=(\delta w+\Delta\delta w)\cos\left(\phi+\frac{d\theta}{2}\right)\quad\textbf{(35)}$$

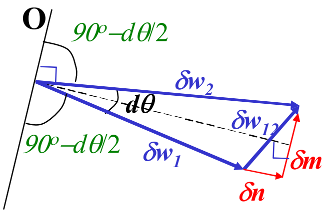
Since,

$$d\theta\approx 0,\cos\frac{d\theta}{2}\approx 1\quad and\quad \sin\frac{d\theta}{2}\approx\frac{d\theta}{2}$$

$$d\delta w=\delta w\, d\theta\tan\phi\longrightarrow\delta w=\delta w_0\exp(\theta\tan\phi)\quad\textbf{(36)}$$

Additional explanation on displacement diagram of log-spiral No.2





$$\delta w_n=\delta w(\theta)$$

from right figure,

$$\delta w_1=\delta w_0(1+d\theta\tan\phi)$$
$$\delta w_2=\delta w_1(1+d\theta\tan\phi)$$
$$\delta w_n=\delta w_{n-1}(1+d\theta\tan\phi)\quad\textbf{(37)}$$

$$\delta w_n=\delta w_0(1+d\theta\tan\phi)^n=\delta w_0\left(1+\frac{\theta\tan\phi}{n}\right)^n\quad\textbf{(38)}$$

$$\delta w_n=\delta w_0\exp(\theta\tan\phi)\quad at\quad n\to\infty,\quad\textbf{(39)}$$

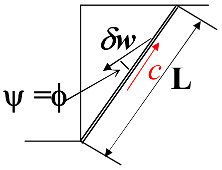
Internal energy dissipation

Internal energy dissipation of ϕ_u=0 (and c_u) materials can be given by the following equations for straight and curved (fan) failure planes respectively.

$$\delta W = \sum c_u \cdot L \cdot \delta w \quad (4.41) \qquad \delta W = 2c_u R \Delta\theta \delta w \quad (4.47)$$

The internal dissipations of c-ϕ materials for both straight and curved (fan) failure planes are derived as follows.

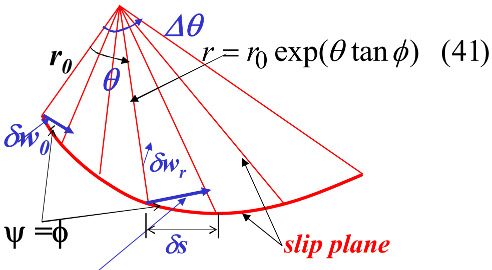
Straight slip plane



Since cohesion, *c*, acts in the direction of slip plane,relative displacement which should be multiplied by *c* in the calculation of the internal dissipation is the component of *δw* in this direction: *δw*conϕ. For the friction term in the material (ϕ=ψ), internal dissipation is zero. Hence internal dissipation on the straight slip plane of c-ϕ material is

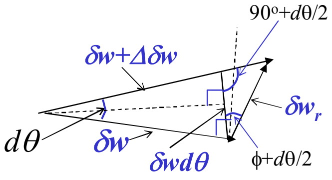
$$\delta W = \sum c \cdot L \cdot \delta w \cos\phi \qquad (40)$$

For fan



$$r = r_0 \exp(\theta \tan\phi) \quad (41)$$

$$\delta w = \delta w_0 \exp(\theta \tan\phi) \quad (42)$$



$$\frac{\delta w_r}{\sin(90^\circ + d\theta/2)} = \frac{\delta w d\theta}{\sin(90 - (\phi + d\theta))}$$

$$at \quad d\theta \to 0,$$

$$\delta w_r \cos\phi \approx \delta w d\theta \quad (43)$$

For radial slip planes:

Using eq.(4) and integrating the eq.(1) in the whole radial slip planes, $d\theta$=>0

$$\Delta W_r = c\int_0^{\Delta\theta} r\cdot\delta w_r \cos\phi = c\int r\cdot\delta w d\theta \quad (44) \longleftarrow \quad \text{eqs.(41) \&(42)}$$

$$= c\int_0^{\Delta\theta} r_0 \exp(\theta\tan\phi)\cdot\delta w_0 \exp(\theta\tan\phi) d\theta$$

$$\Delta W_r = \frac{c r_0 \delta w_0 \cot\phi}{2}\left[\exp(2\Delta\theta\tan\phi)-1\right] \quad (45)$$

For arc slip plane:

$$\Delta W_a = c\int \delta s\cdot\delta w\cos\phi = c\int r\cdot\delta w d\theta \quad (46) \longleftarrow \quad at \quad d\theta\to 0,$$

$$\delta s\cos\phi \approx r d\theta$$

$$\Delta W_a = \frac{c r_0 \delta w_0 \cot\phi}{2}\left[\exp(2\Delta\theta\tan\phi)-1\right] \quad (47)$$

From eqs.(45) & (47), internal dissipation in the fan

$$\underline{\Delta W_f = \Delta W_r + \Delta W_a = c r_0 \delta w_0 \cot\phi\left[\exp(2\Delta\theta\tan\phi)-1\right] \quad (48)}$$

Internal dissipation in discontinuity in 2D

	$\tau_f = c_u$ **(ϕ_u=0 material)**	**$\tau_f = c+\sigma\tan\phi$** **(c-$\phi$ material)**	**remark**
straight line	$\delta W = c_u\cdot L\cdot\delta w$ **(4.41)**	$\delta W = c\cdot L\cdot\delta w\cos\phi$ **(40)**	L: length of slip line δw: relative displacement
curved line (fan)	**circular arc**	**logarithmic spiral**	R: radius of circle r_0: length of radial line at θ=0 δw_0: relative disp. at θ=0 Δθ: radian angle of fan portion
	$\delta W = 2c_u R\Delta\theta\delta w$ **(4.47)**	$\Delta E_f = c r_0\delta w_0 \cot\phi \times\left[\exp(2\Delta\theta\tan\phi)-1\right]$ **(48)**	

Home work 2 -1–individual-
Due date 8th of Nov..

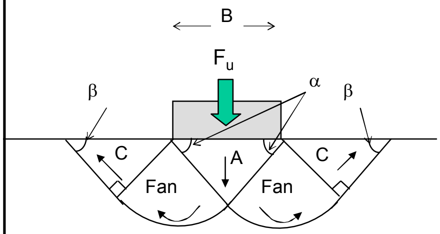


Fig.1

1. Bearing capacity of clay (c_u, ϕ_u=0)

Using failure mechanism shown in Fig.1, which has symmetrical deformation pattern, obtain upper bound of baring capacity by the following sequence.

(1) Draw the displacement diagram of this mechanism.

(2)Estimate the internal energy dissipation.

(3)Derive the baring capacity equation as a function of α and β.

(4)Obtained minimum value of bearing capacity and compare it with the bearing capacity given in page 171and 172 (Foundations and slopes) for the non-symmetrical failure mechanism.

Home work 2-2 –individual-
Due date 8th of Nov.

2. Bearing capacity of sand (ϕ'=ψ,c=0) with weightless material (γ=0)

Using failure mechanism shown in Fig.2, which has symmetrical deformation pattern, obtain upper bound of the bearing capacity by the following sequence.

(1) Draw the displacement diagram of this mechanism.

(2) the external work done done by boundary forces.

(3) Obtain the bearing capacity and compare it with the bearing capacity given in page 217 in the text (Foundations and slopes) for the non-symmetrical failure mechanism.

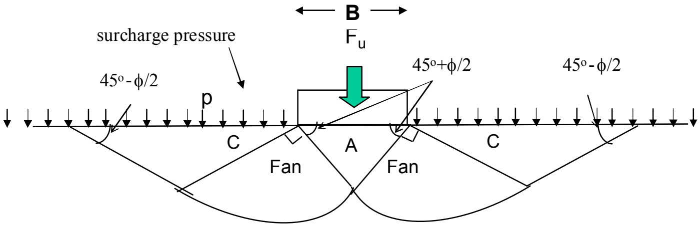


Fig.2

Home work 3
due date ??th of Nov

Select one stability problem and produce a stability chart using upper bound method. See example of slope failure with the failure mechanism shown in Fig.1 and a stability chart on the slope with c,ϕ material as shown in Fig.2.

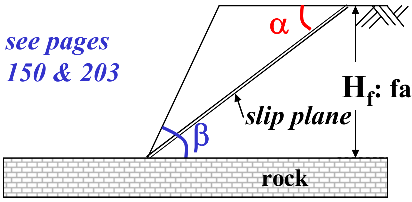


Fig.1 A failure mechanism of slope with uniform c, ϕ and γ

β: slope angle

α: variable to express the failure mechanism (here angle of straight slip plane to the horizon)

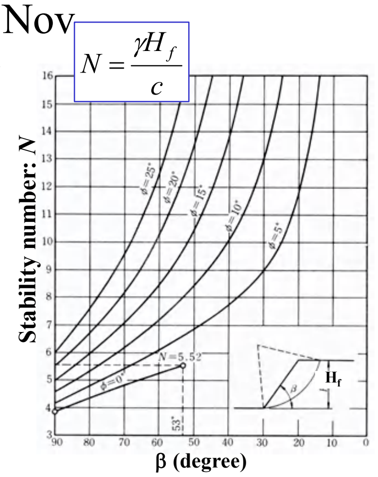
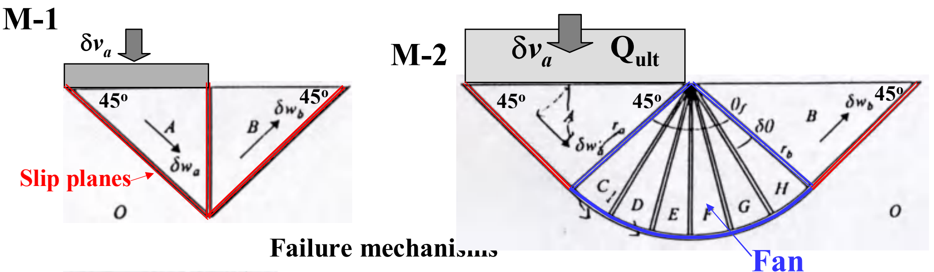
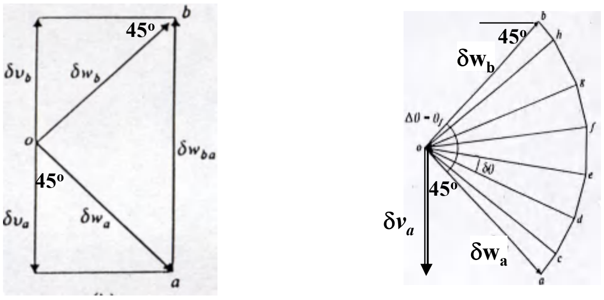


Fig.2 Taylor's stability chart of slope with c-ϕ material obtained by frictional circle method.

Upper bound calculation of bearing capacity: Undrained

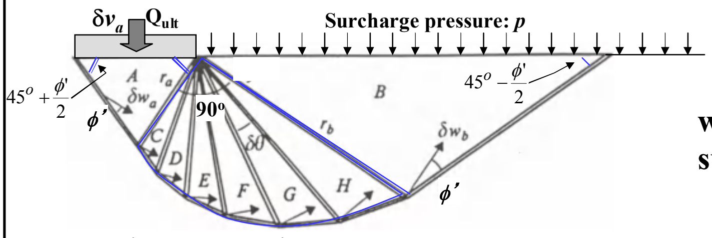


Failure mechanisms



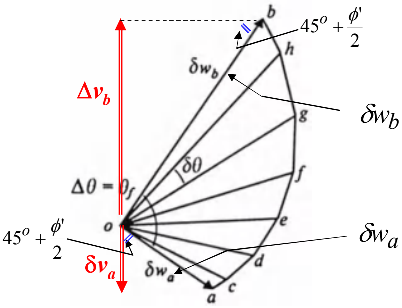
Displacement diagrams

Upper bound calculation of bearing capacity: Drained



Failure mechanism

weightless: γ=0, surcharge: p



$$\delta w_b = \delta w_a \exp\left(\frac{\pi}{2}\tan\phi'\right)$$

$$\delta w_a = \frac{\delta v_a}{\cos\left(45^o + \frac{\phi'}{2}\right)}$$

Displacement diagram